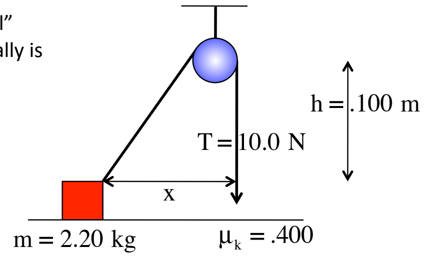
Problem 5.73

(This is an “added thrill” problem because it really is kind of thrilling.)

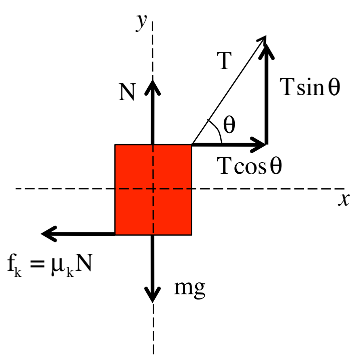
The systems is shown to the right. I’ve assuming the pulley is very small so that the vertical line and the angled line are essentially coming from the same point.

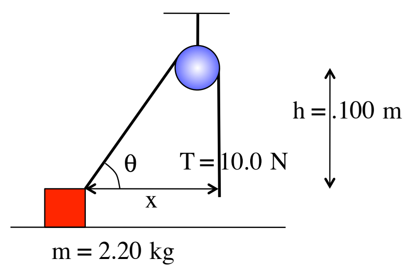


a.) The problem originally asks for the acceleration when x = .400 meters. It is oh so much more fun to determine the acceleration in general as a function of “x,” so that’s what we are going to do (we will put in x = .400 meters later to actually answer the question).

This is one of those problems that is best done using the Formal approach. That’s the way we will do it, again (as usual), following the steps but not enumerating them.

f.b.d. on the mass:



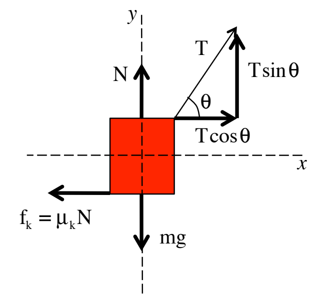


$\sum F_y$:

$$N - mg + T\sin\theta = m\cancel{a_y}^{\,0}$$

$$\Rightarrow \quad N = mg - T\sin\theta$$

How cool! A situation in which N isn’t = mg!



$\sum F_x$:

$$-\mu_k N + T\cos\theta = ma$$

$$\Rightarrow \quad -\mu_k\left(mg - T\sin\theta\right) + T\cos\theta = ma$$

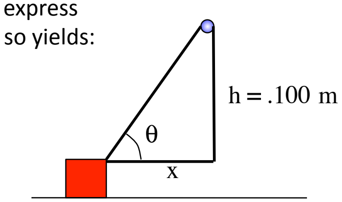
$$\Rightarrow \quad a = \frac{-\mu_k\left(mg - T\sin\theta\right) + T\cos\theta}{m}$$

$$= \frac{-(.400)\left[(2.20\text{ kg})(9.80\text{ m/s}^2) - (10.0\text{ N})\sin\theta\right] + (10.0\text{ N})\cos\theta}{(2.20\text{ kg})}$$

$$= \frac{-(8.62\text{ N}) + (4.00\text{ N})\sin\theta + (10.0\text{ N})\cos\theta}{(2.20\text{ kg})}$$

$$= -(3.92\text{ N}) + (1.82\text{ N})\sin\theta + (4.55\text{ N})\cos\theta$$

To get this into terms of “x,” we need to express the trig functions in terms of “x.” Doing so yields:



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{x}{\sqrt{x^2 + (.100)^2}}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{.100}{\sqrt{x^2 + (.100)^2}}$$

Using our sine and cosine substitutions, the acceleration becomes:

$$a = -(3.92\ \text{N}) + (1.82\ \text{N})\sin\theta + (4.55\ \text{N})\cos\theta$$
$$= -(3.92\ \text{N}) + (1.82\ \text{N})\left(\frac{.100}{\sqrt{x^2+(.100)^2}}\right) + (4.55\ \text{N})\left(\frac{x}{\sqrt{x^2+(.100)^2}}\right)$$

Rewriting (and eliminating the units) for easier viewing:

$$a = -3.92 + .182\left(x^2+(.100)^2\right)^{-1/2} + 4.55x\left(x^2+(.100)^2\right)^{-1/2}$$

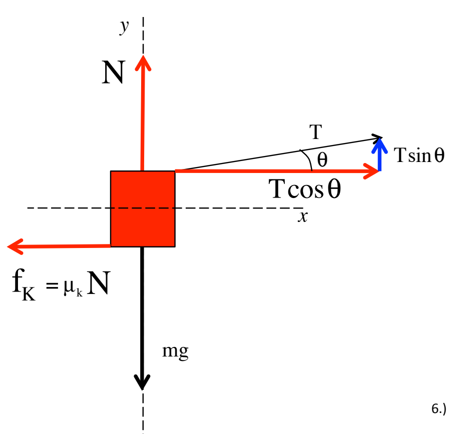
So what's the acceleration when x = .400 meters?

$$a = -3.92 + .182\left(x^2+(.100)^2\right)^{-1/2} + 4.55x\left(x^2+(.100)^2\right)^{-1/2}$$
$$= -3.92 + .182\left((.400)^2+(.100)^2\right)^{-1/2} + 4.55(.400)\left((.400)^2+(.100)^2\right)^{-1/2}$$
$$= .932\ \text{m/s}^2$$

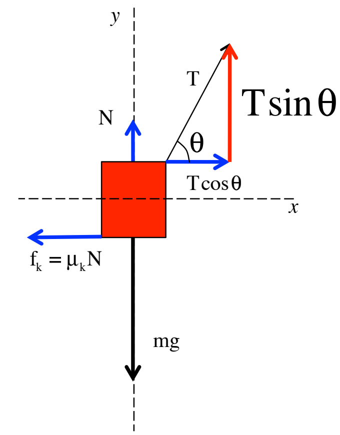
b.) How does "a" vary as the block slides from far away to x = 0?

This is an interesting problem conceptually because there is a lot going on. The short answer is that you'd expect the body to pick up speed when the tension was mainly horizontal and begin to slow down as the tension got close to the vertical (look back at the sketch).

Answering more completely and in pieces: When the rope is fairly oblique (large "x", small angle), the tension component in the *normal direction* will be fairly small which means the normal force will need to be fairly large (the sum of the two has to counteract "mg") and the associated frictional force will be fairly large. The accelerating component of the tension will also be large, though, so one might expect the overall acceleration to be positive.



When the block gets closer (small "x", large angle), the tension component in the *normal direction* will be fairly large which means the normal force will be fairly small (again, the two have to add to the constant "mg") and the associated frictional force will be fairly small. The accelerating component of the tension will also be small with the net effect being that friction will win out and the body will slow under the influence of a negative acceleration.



And, of course, somewhere in-between the net accelerating force will be zero as the acceleration transits from being positive to being negative.

Specifically, though, where will the net acceleration be zero? Let's see what the main suggests. We have two relationships, one in "θ" and one in "x." We will look first at the angular relationship:

$$a = -(3.92\ \text{N}) + (1.82\ \text{N})\sin\theta + (4.55\ \text{N})\cos\theta$$

For large x's, the angle will be small. As cosine dominates at small angles ($\cos 0^\circ = 1$, which is as small an angle and big a magnitude as one can get), the acceleration will be positive. That is, the "$(4.55\ \text{N})\cos\theta$" term will be larger than the -3.92 N term, with the "$(1.82\ \text{N})\sin\theta$" term helping out a little. When the "x" value is small, then angle will be large and the sine function will dominate ($\sin 90^\circ = 1$, which again is as large a magnitude as it gets). But the "$(1.82\ \text{N})\sin\theta$" term with a little help from the "$(4.55\ \text{N})\cos\theta$" term, isn't going to be big enough to overpower the -3.92 N term, so in close the acceleration is going to be negative and the block will be slowing down.

And lastly, from the "x" perspective:

$$a=-(3.92\text{ N})+(1.82\text{ N})\left(\frac{.100}{\sqrt{x^2+(.100)^2}}\right)+(4.55\text{ N})\left(\frac{x}{\sqrt{x^2+(.100)^2}}\right)$$

For large x's, the "4.55x" term is going to govern, with a little help from the "1.82" term, and the two are going to be larger than the "-3.92 N" term. In that region, the acceleration will be positive. As "x" gets small, the "4.55x" term will diminish in importance and the "1.82" term will come into play with a little help from the "4.55x" term. Unfortunately, it won't be enough to counter the "-3.92 N" term and the acceleration will be negative.

c.) Determine the maximum value for the acceleration.

This is where the fun begins. The maximum acceleration will occur at the acceleration function's turn-around point, or where its derivative is zero. That information was asked for in "x," though we will do it for both just because it's fun. Here goes:

Determine the "x" coordinate at which the maximum acceleration happens:

a_{max} happens when $\frac{da}{dx}=0$, so:

$$\frac{da}{dx}=\frac{d\left(-(3.92\text{ N})+(.182\text{ N})\left(x^2+(.100)^2\right)^{-1/2}+(4.55\text{ N})x\left(x^2+(.100)^2\right)^{-1/2}\right)}{dx}$$

$$=+\left(-\frac{1}{2}\right)(2x)(.182\text{ N})\left(x^2+(.100)^2\right)^{-3/2}+(4.55\text{ N})\left(x^2+(.100)^2\right)^{-1/2}$$

$$+(4.55\text{ N})x\left(-\frac{1}{2}\right)(2x)\left(x^2+(.100)^2\right)^{-3/2}$$

$$=-x(.182\text{ N})\left(x^2+(.100)^2\right)^{-3/2}+(4.55\text{ N})\left(x^2+(.100)^2\right)^{-1/2}-(4.55\text{ N})x^2\left(x^2+(.100)^2\right)^{-3/2}$$

The acceleration will be a maximum when this expression is evaluated at zero, or

$$-x(.182\text{ N})\left(x^2+(.100)^2\right)^{-3/2}+(4.55\text{ N})\left(x^2+(.100)^2\right)^{-1/2}-(4.55\text{ N})x^2\left(x^2+(.100)^2\right)^{-3/2}=0$$

Solving:

$$-x(.182\text{ N})\left(x^2+(.100)^2\right)^{-3/2}+(4.55\text{ N})\left(x^2+(.100)^2\right)^{-1/2}-(4.55\text{ N})x^2\left(x^2+(.100)^2\right)^{-3/2}=0$$

$$\Rightarrow\quad -\frac{x(.182\text{ N})}{\left(x^2+(.100)^2\right)}+(4.55\text{ N})-\frac{(4.55\text{ N})x^2}{\left(x^2+(.100)^2\right)}=0$$

$$\Rightarrow\quad x(.182\text{ N})+(4.55\text{ N})-\left(x^2+(.100)^2\right)(4.55\text{ N})x^2=0$$

$$\Rightarrow\quad .182x+4.55x^2+4.55(.100)^2-4.55x^2=0$$

$$\Rightarrow\quad .182x+4.55(.100)^2=0$$

$$\Rightarrow\quad x=.250$$

Lots of obscure math, but it can be done!

In any case, assuming you have "x," you can get "a" as follows:

$$a_{max}=-(3.92\text{ N})+(1.82\text{ N})\left(\frac{.100}{\sqrt{x^2+(.100)^2}}\right)+(4.55\text{ N})\left(\frac{x}{\sqrt{x^2+(.100)^2}}\right)$$

$$=-(3.92\text{ N})+(1.82\text{ N})\left(\frac{.100}{\sqrt{(.250\text{ m})^2+(.100)^2}}\right)+(4.55\text{ N})\left(\frac{(.250\text{ m})}{\sqrt{(.250\text{ m})^2+(.100)^2}}\right)$$

$$=.976\text{ m/s}^2$$

So how might we have done this using angles? I promised that, too, so here goes:

a_{max} happens when $\frac{da}{d\theta}=0$, so:

$$\frac{da}{d\theta} = \frac{d\left(-(3.92\text{ N}) + (1.82\text{ N})\sin\theta + (4.55\text{ N})\cos\theta\right)}{d\theta} = 0$$

$$\Rightarrow \quad 1.82\cos\theta - 4.55\sin\theta = 0$$

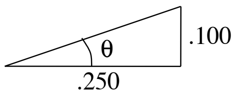
$$\Rightarrow \quad 1.82\cos\theta = 4.55\sin\theta$$

$$\Rightarrow \quad \frac{1.82}{4.55} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1.82}{4.55}\right)$$

$$= 21.8^\circ$$

Seems a lot easier . . . In any case, a quick way to see if it checks out is to determine the angle we'd get if x = .250 meters (as determined in the previous section). In that case:



$$\theta = \tan^{-1}\left(\frac{.100}{.250}\right)$$

$$= 21.8^\circ$$

It checks! Yahoooooo . . .

d.) Where will the acceleration be zero?

This should be easy!

$$a = -3.92 + \frac{.182}{\sqrt{x^2 + (.100)^2}} + \frac{4.55x}{\sqrt{x^2 + (.100)^2}} = 0$$

$$\Rightarrow \quad -3.92\sqrt{x^2 + (.100)^2} + .182 + 4.55x = 0$$

$$\Rightarrow \quad 3.92\sqrt{x^2 + (.100)^2} = .182 + 4.55x$$

$$\Rightarrow \quad \left(3.92\sqrt{x^2 + (.100)^2}\right)^2 = (.182 + 4.55x)^2$$

$$\Rightarrow \quad 3.92^2\left(x^2 + (.100)^2\right) = .0331 + 1.66x + 20.7x^2$$

$$\Rightarrow \quad 5.30x^2 - 1.66x - .121 = 0$$

$$\Rightarrow \quad x = .0610\text{ m}$$

Bawdaa bawdaa bawdaa's all folks!